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Uniqueness of meromorphic functions with their derivatives sharing a small function

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Abstract

In this paper, we investigate the problems concerning meromorphic functions sharing a small function with their derivatives. We study the uniqueness of meromorphic functions of the form and using the notion of weighted sharing

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1 Introduction and main results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [4]. Let $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a IM (ignoring multiplicity) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicity).

A function $a(z)$ is said to be a small function of f, if $a(z)$ is a meromorphic function satisfying $T(r, a(z)) = S(r, f)$, i.e. $T(r, a) = o(T(r, f))$ as $r \to +\infty$, possibly outside a set of finite linear measure. We define $E(a, f) = \{z \in \mathbb{C} : f(z) - a(z) = 0\}$ where a zero of $f - a$ is counted according to its multiplicity, similarly $\overline{E}(a, f)$ denotes the zeros of $f - a$, where a zero is counted only once. For a non-negative integer k, we denote by $E_k(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, then f and g share the function a with weight k.

We write "f and g share (a, k)" to mean that "f and g share the function a with weight k". If f and g share (a, k), then f and g share (a, p) for $0 \leq p < k$.

For notational purposes, let f and g share 1 IM. Let z_0 be a 1-point of f of order p, a 1-point of g of order q. We denote by $N_{11}\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of f and g where $p = q = 1$. By $N_E^{(2}\left(r, \frac{1}{f-1}\right)$ we denote the counting function of those 1-points of f and g where $p = q \geq 2$. Also, $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ denotes the counting function of those 1-points of both f and g where $p > q$. It is easy to see that

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f-1}\right) &= N_{11}\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{g-1}\right) + N_E^{(2}\left(r, \frac{1}{g-1}\right)\\ &= \overline{N}\left(r, \frac{1}{g-1}\right)\end{aligned}$$

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For a positive integer k and $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ $\left(or\ \overline{N}_{k)}\left(r, \frac{1}{f-a}\right)\right)$ the counting function (reduced counting function) of those $a-$points of f whose multiplicities are not less than p. Similarly, $N_{(k}\left(r, \frac{1}{f-a}\right)$ $\left(or\ \overline{N}_{(k}\left(r, \frac{1}{f-a}\right)\right)$ the counting function (reduced counting function) of those $a-$points of f whose multiplicities are not greater than p.
Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right),$$

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

$$\delta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Clearly,

$$0 \le \delta(a, f) \le \delta_k(a, f) \le \delta_{k-1}(a, f)... \le \delta_1(a, f) = \Theta(a, f).$$

In 1996, Brück [3] proposed the following famous conjecture.
Conjecture. Let f be a non-constant entire function. Suppose

$$\rho_1(f) = \limsup_{r\to\infty} \frac{log\, log\, T(r, f)}{log\, r}.$$

If $\rho_1(f)$ is not a positive integer or infinite and if f and f' share the value 1 CM, then

$$\frac{f'-1}{f-1} \equiv c \text{ for some non-zero constant } c.$$

Regarding the above conjecture, investigations and many results have been obtained (see. [5], [7], [8]).

In 2005, Zhang [9] studied the problem of a meromorphic function sharing a small function and obtained the following result.

Theorem A. Let f be a non-constant meromorphic function and $k(\ge 1)$, $l(\ge 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \ge 2$ and

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4, \tag{1.1}$$

or if $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6, \tag{1.2}$$

or if $l = 0$ and

$$(6+2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10, \tag{1.3}$$

then $f \equiv f^{(k)}$.

Recently, J. D. Li [6], improved the above result by replacing the conditions in (1.1) - (1.3) by weaker ones and obtained the following result.

Theorem B. Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and

$$(3+k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 4,$$

or if $l = 1$ and

$$\left(\frac{7}{2} + k\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 5,$$

or if $l = 0$ and

$$(6+2k)\Theta(\infty, f) + 2\Theta(0, f) + \delta_2(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10,$$

then $f \equiv f^{(k)}$.

To state our main result, we assume the following notations.

Let $\mathcal{P}(w) = a_{n+m}w^{n+m} + ... + a_n w^n + ... + a_0 = a_{n+m}\prod_{i=1}^{s}(w - w_{p_i})^{p_i}$,

where $a_j (j = 0, 1, 2, ..., n + m - 1)$, $a_{n+m} \neq 0$ and $w_{p_i}(i = 1, 2, ..., s)$ are distinct finite complex numbers and $2 \leq s \leq n + m$ and $p_1, p_2, ..., p_s$, $s \geq 2$, n, m and k are all positive integers with $\sum_{i=1}^{s} p_i = n + m$.

Let $p > \max_{p \neq p_i, i=1,2,...,r}\{p_i\}$, $r = s - 1$, where s and r are two positive integers.

Let $P(w_1) = a_{n+m}\prod_{i=1}^{s-1}(w_1 + w_p - w_{p_i})^{p_i} = b_q w_1^q + b_{q-1}w_1^{q-1} + ... + b_0$, where $a_{n+m} = b_q$, $w_1 = w - w_p$, $q = n + m - p$. Therefore, $\mathcal{P}(w) = w_1^p P(w_1)$.

We assume $P(w_1) = b_q \prod_{i=1}^{r}(w_1 - \alpha_i)^{p_i}$, where $\alpha_i = w_{p_i} - w_p$, $(i = 1, 2, ..., r)$, be distinct zeros of $P(w_1)$.

Definition 1.1 (see [2]). For two positive integers n, p we define $\mu_p = min\{n, p\}$ and $\mu_p^* = p + 1 - \mu_p$. Then it is clear that

$$N_p\left(r, \frac{1}{f^n}\right) \leq \mu_p N_{\mu_p^*}\left(r, \frac{1}{f}\right). \tag{1.4}$$

In the present paper, we extend Theorem B by investigating the uniqueness of meromorphic functions of the form $f_1^p P(f_1) - a$ and $(f_1^p P(f_1))^{(k)} - a$ and obtain the following result.

Theorem 1.1. Let $k(\geq 1)$, $l(\geq 0)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers, f and $f_1 = f - w_p$ be two non-constant meromorphic functions. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + ... + a_n z^n + ... + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function with respect to f. Suppose $\mathcal{P}(f) - a$ and $(\mathcal{P}(f))^{(k)} - a$ share $(0, l)$.
If $l \geq 2$ and

$$(k+3)\Theta(\infty, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > m + n - 2p + k + 3 + \mu_2 + \mu_{k+2} \tag{1.5}$$

or $l = 1$ and

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > \frac{3(m+n) - 5p}{2} + k + 4 + \mu_2 + \mu_{k+2} \tag{1.6}$$

or $l = 0$ and

$$(2k+6)\,\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+1}\delta_{\mu_{k+1}^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > 4(m+n) - 5p + 2k + 8 + \mu_2 + \mu_{k+1} + \mu_{k+2} \tag{1.7}$$

then $\mathcal{P}(f) \equiv (\mathcal{P}(f))^{(k)}$.

We can easily deduce the following corollaries from the above theorem.

Corollary 1.2. Let $k(\geq 1)$, $l(\geq 0)$, $n(\geq 1)$, $p(\geq 1)$ and $m(\geq 0)$ be integers, f and $f_1 = f - w_p$ be two non-constant entire functions. Let $\mathcal{P}(z) = a_{m+n}z^{m+n} + ... + a_n z^n + ... + a_0$, $a_{m+n} \neq 0$, be a polynomial in z of degree $m+n$ such that $\mathcal{P}(f) = f_1^p P(f_1)$. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Suppose $\mathcal{P}(f) - a$ and $(\mathcal{P}(f))^{(k)} - a$ share $(0, l)$.
If $l \geq 2$ and

$$\delta_{\mu_{k+2}^*}(w_p, f) > 1 + \frac{m+n-2p}{\mu_2 + \mu_{k+2}} - \frac{\mu_2}{\mu_2 + \mu_{k+2}}\left[\delta_{\mu_2^*}(w_p, f) - \delta_{\mu_{k+2}^*}(w_p, f)\right]$$

or $l = 1$ and

$$\delta_{\mu_{k+2}^*}(w_p, f) > 1 + \frac{3(m+n) - 5p}{2(\mu_2 + \mu_{k+2}) + 1} - \frac{1}{2(\mu_2 + \mu_{k+2}) + 1}\left[2\mu_2\delta_{\mu_2^*}(w_p, f) - (2\mu_2 + 1)\delta_{\mu_{k+2}^*}(w_p, f) - \Theta(w_p, f)\right]$$

or $l = 0$ and

$$\delta_{\mu_{k+2}^*}(w_p, f) > 1 + \frac{4(m+n) - 5p}{\mu_2 + \mu_{k+1} + \mu_{k+2} + 2} - \frac{1}{\mu_2 + \mu_{k+1} + \mu_{k+2} + 2}\left[\mu_2\delta_{\mu_2^*}(w_p, f) + \mu_{k+1}\delta_{\mu_{k+1}^*}(w_p, f) - (2 + \mu_2 + \mu_{k+2})\delta_{\mu_{k+2}^*}(w_p, f) + 2\Theta(w_p, f)\right]$$

then $\mathcal{P}(f) \equiv (\mathcal{P}(f))^{(k)}$.

Corollary 1.3. Let $k(\geq 1)$, $l(\geq 0)$, $n(\geq 1)$ and $m(\geq 0)$ be integers, f be non-constant meromorphic function. Let $P(z) = a_m z^m + ... + a_0$, $a_m \neq 0$, be a polynomial in z of degree m. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose $f^n P(f) - a$ and $(f^n P(f))^{(k)} - a$ share $(0, l)$.
If $l \geq 2$ and

$$(k+3)\Theta(\infty, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > m - p + k + 3 + \mu_2 + \mu_{k+2}$$

or $l = 1$ and

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \frac{1}{2}\Theta(w_p, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > \frac{3m - 2p}{2} + k + 4 + \mu_2 + \mu_{k+2}$$

or $l = 0$ and

$$(2k+6)\,\Theta(\infty, f) + 2\Theta(w_p, f) + \mu_2 \delta_{\mu_2^*}(w_p, f) + \mu_{k+1}\delta_{\mu_{k+1}^*}(w_p, f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p, f) > 4m - p + 2k + 8 + \mu_2 + \mu_{k+1} + \mu_{k+2}$$

then $f^n P(f) \equiv (f^n P(f))^{(k)}$.

The following example shows that the conditions in (1.5) - (1.7) in Theorem 1.1 cannot be removed.

Example 1. Let $f(z) = cos(\alpha z) + a - \frac{a}{\alpha^{8d}}, d \in N$; where $\alpha \neq 0, \alpha^{8d} \neq 1$ and $a \in \mathbb{C} - \{0\}$. Let $p = n = 1, w_p = 0$ *and* $m = 0$. Let $\mathcal{P}(f) = f$, $\mathcal{P}(f)^{(k)} = f^{(8d)}$. Then $\mathcal{P}(f)^{(k)} = cos(\alpha z)\alpha^{8d}$. Here $m = 0$, $\mu_2 = 1$. Again $\Theta(\infty; f) = 1$ and

$$\overline{N}\left(r, \frac{1}{f}\right) = \overline{N}\left(r, \frac{1}{cos(\alpha z) - (a - \frac{a}{\alpha^{8d}})}\right) \sim T(r, f).$$

Therefore,

$$\Theta(0; f) = 0 = \delta_p(0; f), \forall p \in N.$$

Also it is clear that $\mathcal{P}(f)$ and $\mathcal{P}(f)^{(k)}$ share (a, l) $(l \geq 0)$ but none of the inequalities (1.5), (1.6) and (1.7) of Theorem 1.1 is satisfied and $\mathcal{P}(f) \not\equiv \mathcal{P}(f)^{(k)}$.

2 Preliminary Lemmas

Let F and G be two non-constant meromorphic functions. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 (see [9]). Let f be a non constant meromorphic function, k, p, be two positive integers, then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

$$\text{Clearly, } \overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right).$$

Lemma 2.2 (see [6]). Let H be defined as in (2.1). If F and G share 1 IM and $H \not\equiv 0$, then

$$N_{11}\left(r, \frac{1}{F-1}\right) \le N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.3 (see [1]). Let F and G share $(1,l)$ and $\overline{N}(r,F) = \overline{N}(r,G)$ and $H \not\equiv 0$, then

$$\begin{aligned} N(r,H) \le \overline{N}(r,F) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) \\ + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r,f). \end{aligned}$$

3 Proof of the Theorem

Proof of Theorem 1. Let $F = \dfrac{\mathcal{P}(f)}{a} = \dfrac{f_1^p P(f_1)}{a}$ and $G = \dfrac{(\mathcal{P}(f))^{(k)}}{a} = \dfrac{(f_1^p P(f_1))^{(k)}}{a}$.
Since $\mathcal{P}(f) - a$ and $[\mathcal{P}(f)]^{(k)} - a$ share $(0,l)$, F, G share $(1,l)$ except the zeros and poles of $a(z)$. Also note that $\overline{N}(r,F) = \overline{N}(r,f) + S(r,f)$ and $\overline{N}(r,G) = \overline{N}(r,f) + S(r,f)$. Let H be defined as in (2.1).
We consider the following cases.

Case 1. Suppose $H \not\equiv 0$. By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) + T(r,G) \le \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ - \overline{N}_0\left(r, \frac{1}{F'}\right) - \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r,F) + S(r,G), \end{aligned} \tag{3.1}$$

where $\overline{N}_0\left(r, \dfrac{1}{F'}\right)$ denotes the reduced counting function of the zeros of F' which are not the zeros of $F(F-1)$.
Since F and G share 1 IM, it is easy to verify that

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F-1}\right) = N_{11}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) \\ + N_E^{(2}\left(r, \frac{1}{G-1}\right) = \overline{N}\left(r, \frac{1}{G-1}\right). \end{aligned} \tag{3.2}$$

Using Lemmas 2.2, 2.3, (3.1) and (3.2), we get

$$\begin{aligned} T(r,F) + T(r,G) \le 3\overline{N}(r,F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_{11}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{G-1}\right) \\ + 3\overline{N}_L\left(r, \frac{1}{F-1}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned} \tag{3.3}$$

Subcase 1.1. Let $l \geq 2$.
Obviously,

$$N_{11}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{G-1}\right) + 3\overline{N}_L\left(r, \frac{1}{F-1}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right)$$
$$\leq N\left(r, \frac{1}{G-1}\right) + S(r,F)$$
$$\leq T(r,G) + S(r,F) + S(r,G). \tag{3.4}$$

Using (3.3) and (3.4), we get

$$T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r,F) + S(r,F). \tag{3.5}$$

Using Lemma 2.1, (1.4) and (3.5), we get

$$(n+m)T(r,f) \leq N_2\left(r, \frac{1}{f_1^p P(f_1)}\right) + N_2\left(r, \frac{1}{(f_1^p P(f_1))^{(k)}}\right) + 3\overline{N}(r,f) + S(r,f)$$
$$\leq 3\overline{N}(r,f) + \mu_2 N_{\mu_2^*}\left(r, \frac{1}{f-w_p}\right) + (n+m-p)T(r,f)$$
$$+ N_{k+2}\left(r, \frac{1}{f_1^p P(f_1)}\right) + k\overline{N}(r,f) + S(r,f)$$
$$\leq (k+3)\overline{N}(r,f) + \mu_2 N_{\mu_2^*}\left(r, \frac{1}{f-w_p}\right) + 2(n+m-p)T(r,f)$$
$$+ \mu_{k+2} N_{\mu_{k+2}^*}\left(r, \frac{1}{f-w_p}\right) + S(r,f).$$

So, $(k+3)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(w_p,f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f) \leq m+n-2p+k+3+\mu_2+\mu_{k+2}$,

which contradicts with (1.5).

Subcase 1.2. Let $l = 1$.
It is easy to verify that

$$N_{11}\left(r, \frac{1}{F-1}\right) + 2N_E^{(2}\left(r, \frac{1}{G-1}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right)$$
$$\leq N\left(r, \frac{1}{G-1}\right) + S(r,F)$$
$$\leq T(r,G) + S(r,F) + S(r,G). \tag{3.6}$$

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2}N\left(r, \frac{F}{F'}\right)$$
$$\leq \frac{1}{2}N\left(r, \frac{F'}{F}\right) + S(r,F)$$
$$\leq \frac{1}{2}\left(\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r,F)\right) + S(r,F). \tag{3.7}$$

Using (3.3), (3.6) and (3.7), we get

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \frac{7}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + S(r,F). \tag{3.8}$$

Using Lemma (2.1), (1.4) and (3.8), we get

$$\begin{aligned}(n+m)T(r,f) \leq &\left(k+\frac{7}{2}\right)\overline{N}(r,f) + \mu_2 N_{\mu_2^*}\left(r,\frac{1}{f-w_p}\right) + \mu_{k+2}N_{\mu_{k+2}^*}\left(r,\frac{1}{f-w_p}\right)\\ &+\frac{1}{2}\overline{N}\left(r,\frac{1}{f-w_p}\right) + \frac{5}{2}(n+m-p)T(r,f) + S(r,f).\end{aligned}$$

$$\text{So,}\quad \left(k+\frac{7}{2}\right)\Theta(\infty,f) + \mu_2\delta_{\mu_2^*}(w_p,f) + \mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f) + \frac{1}{2}\Theta(\infty,f)$$
$$\leq \frac{3(m+n)-5p}{2} + k + 4 + \mu_2 + \mu_{k+2}.$$

which contradicts with (1.6).

Subcase 1.3. Let $l = 0$.
It is easy to verify that

$$\begin{aligned}N_{11}\left(r,\frac{1}{F-1}\right) &+ 2N_E^{(2}\left(r,\frac{1}{G-1}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right) + S(r,F)\\ &\leq T(r,G) + S(r,F) + S(r,G).\end{aligned} \tag{3.9}$$

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{F-1}\right) &\leq N\left(r,\frac{1}{F-1}\right) - \overline{N}\left(r,\frac{1}{F-1}\right)\\ &\leq N\left(r,\frac{F}{F'}\right) \leq N\left(r,\frac{F'}{F}\right) + S(r,F)\\ &\leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,F).\end{aligned} \tag{3.10}$$

Using (3.3), (3.9) and (3.10), we get

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 6\overline{N}(r,F) + 2\overline{N}\left(r,\frac{1}{F}\right) + N_1\left(r,\frac{1}{G}\right) + S(r,F). \tag{3.11}$$

Using Lemma 2.1 and (3.11), we get

$$\begin{aligned}(n+m)T(r,f) \leq\; &N_2\left(r,\frac{1}{f_1^pP(f_1)}\right) + N_2\left(r,\frac{1}{(f^pP(f_1))^{(k)}}\right) + 6\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f_1^pP(f_1)}\right)\\ &+ N_1\left(r,\frac{1}{(f_1^pP(f_1))^{(k)}}\right) + S(r,f).\end{aligned}$$

So,

$$\text{So,}\quad (2k+6)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\mu_{k+1}\delta_{\mu_{k+1}^*}(w_p,f)+\mu_{k+2}\delta_{\mu_{k+2}^*}(w_p,f) \leq 4(m+n)-5p+2k+8+\mu_2+\mu_{k+1}+\mu_{k+2}.$$

which contradicts with (1.7).

Case 2. Suppose $H \equiv 0$. Using (2.1), we get

$$\frac{F''}{F'}-\frac{2F'}{F-1}=\frac{G''}{G'}-\frac{2G'}{G-1}.$$

Hence,

$$\frac{1}{F-1}\equiv C\frac{1}{G-1}+D, \tag{3.12}$$

where C, D are constants and $C \neq 0$.
We discuss the following three cases:

Subcase 2.1. When $D \neq 0, -1$.
Rewrite (3.12) as,

$$\frac{G-1}{C}=\frac{F-1}{D+1-DF},$$

we have,

$$\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-\frac{(D+1)}{D}}\right).$$

By using second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}(n+m)T(r,f) &= T(r,F)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{(D+1)}{D}}\right)+S(r,f)\\ &\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,f)\\ &\leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f_1^pP(f_1)}\right)+S(r,f)\\ &\leq 2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-w_p}\right)+(n+m-p)T(r,f)+S(r,f).\end{aligned}$$

$$\text{So,}\quad 2\Theta(\infty,f)+\Theta(w_p,f)\leq 3-p,$$

which contradicts with (1.5), (1.6) and (1.7).

Subcase 2.2. When $D=0$. Then from (3.12), we get

$$G = CF - (C-1). \tag{3.13}$$

If $C \neq 1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \frac{(C-1)}{C}}\right).$$

Proceeding as in Subcase 2.1, we get

$$(n+m)T(r,f) \leq (k+1)\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-w_p}\right) + 2(n+m-p)T(r,f) + N_{k+1}\left(r, \frac{1}{f-w_p}\right) + S(r,f).$$

$$\text{So,}\quad (k+1)\Theta(\infty,f) + \Theta(w_p,f) + \mu_{k+1}\delta_{\mu^*_{k+1}}(w_p,f) \leq k+2+\mu_{k+1}+n+m-2p,$$

which contradicts with (1.5), (1.6) and (1.7).
Therefore, $C=1$.
By using (3.13), we get $F \equiv G$ and so, $f_1^p P(f_1) = (f_1^p P(f_1))^{(k)}$.

Subcase 2.3. When $D=-1$. Then from (3.12) we get

$$\frac{1}{F-1} = \frac{C}{G-1} - 1$$
$$\Rightarrow \frac{F}{F-1} = \frac{C}{G-1}.$$

Hence we have $\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}(r,G) = S(r,f)$ and hence $\overline{N}\left(r, \frac{1}{f}\right) = S(r,f)$.
If $C \neq -1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \dfrac{C}{C+1}}\right).$$

Proceeding as in Subcase 2.1, we get

$$(n+m)T(r,f) \leq (k+1)\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-w_p}\right) + 2(n+m-p)T(r,f) + N_{k+1}\left(r, \frac{1}{f-w_p}\right) + S(r,f).$$

$$\text{So,}\quad (k+1)\Theta(\infty,f) + \Theta(w_p,f) + \mu_{k+1}\delta_{\mu^*_{k+1}}(w_p,f) \leq k+2+\mu_{k+1}+n+m-2p$$

which contradicts with (1.5), (1.6) and (1.7).
Therefore, $C=-1$.
By using (3.13), we get $FG \equiv 1$.
Hence, $\mathcal{P}(f)(\mathcal{P}(f))^{(k)} = a^2$. Thus in this case,

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r,f).$$

Hence we have,

$$\frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)} = \frac{a^2}{(\mathcal{P}(f))^2} \tag{3.14}$$

From first fundamental theorem and (3.14), we get

$$\begin{aligned}
2T(r, \mathcal{P}(f)) &\leq T\left(r, \frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)}\right) \\
&\leq N\left(r, \frac{(\mathcal{P}(f))^{(k)}}{\mathcal{P}(f)}\right) + S(r,f) \\
&\leq k\left(\overline{N}(r, \mathcal{P}(f)) + \overline{N}\left(r, \frac{1}{\mathcal{P}(f)}\right)\right) + S(r,f) \\
&\leq S(r,f),
\end{aligned}$$

which is impossible.
Hence the theorem.

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